Fundamental frequency analysis of functionally graded sandwich beams based on the state space approach

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Abstract

The state space approach is used to provide analytical solution for fundamental frequency analysis of functionally graded sandwich beams. The classical beam theory, first-order and higher-order shear deformation theories are employed to consider beams of various classical and non-classical boundary conditions. Governing equations of motions are derived from Hamilton's principle. The research investigates the effect of boundary conditions on the fundamental frequency with nine combinations of classical boundary conditions created from clamped, hinged, pinned and free conditions in accordance with three combinations of non-classical boundary conditions created from the assumption of an elastic support. In addition, the influence of material parameter and arrangement of layers as well as the slenderness ratio in vibration of functionally graded sandwich beams is examined.

Keywords: Vibration, functionally graded sandwich beams, state space approach, non-classical boundary conditions.

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1. Introduction

Functionally Graded Materials (FGMs) are a class of composite materials in which the mechanical properties vary gradually and continuously from one layer to the other. Such materials are created from the exploitation of basic material elements into various organic and inorganic compounds to produce advanced polymers and elastomers, alloys, glasses and ceramics. The combination of FGMs and sandwich structures, often referred to as FG sandwich structures, has become even more attractive due to the designable material properties and the possibility to eliminate the delamination which occurs in conventional composite structures. Many applications of FGMs can be found in aerospace structures and electronics as well as biomedical installations. Latest research into FGMs and FG sandwich structures involve the expansion of using these structures, simplified approaches to homogenize FGMs and the development of accurate theories and techniques to analyse the behaviour of such structures.

Various shear deformation theories and analysis techniques have been developed to investigate vibration analysis of FG sandwich beams. The Euler-Bernoulli theory known as Classical Beam Theory (CBT), which neglects the shear effect, provides acceptable results for thin beams. This theory was applied to study the dynamic behaviour of FG beams by Alshorbagy et al. [1], Simsek and Kocaturk [2] and Jin and Wang [3]. It is worth noting that, solutions based on the CBT overestimate natural frequencies of moderately thick beams. To overcome the limitation of the CBT, many shear deformation theories have been proposed. The Timoshenko beam theory known as the First-order Beam Theory (FOBT), which is the simplest model considering shear effect, attracts much attention of researchers. Aydogdu et al. [4] presented Navier solution of natural frequencies for FG beams. Su and Banerjee [5, 6] developed dynamic stiffness method for FG beams' free vibration analysis using CBT and FOBT. Nguyen et al. [7] proposed an improvement on FOBT to present Navier solution for static and vibration responses of FG beams under axial load. Sina et al. [8] also applied FOBT to analyse free vibration of FG beams. However, the FOBT needs the shear correction factor to modify the results due to the dissatisfaction of free stress at the faces. This factor is difficult to determine exactly since it depends on many parameters. To overcome this problem, the Higher-order Beam Theory (HOBT) was developed to accurately analyse beam's behaviour without using shear correction factor. Thai and Vo [9] investigated static and free vibration behaviour of FG beams using various HOBTs. Wattanasakulpong [10] and Pradhan and Chakraverty [11, 12] employed Rayleigh-Ritz method using the CBT, FOBT and HOBT to analyse natural vibration of FG beams. Simsek [13] obtained fundamental frequencies of FG beams with varying boundary conditions based on Lagrange approach. Mashat et al. [14] applied Carrera Unified Formulation to investigate free vibration of FG layered beams. Vo et al. [15] and Nguyen et al. [16] also employed different HOBTs with finite element and Rayleigh-Ritz methods to analyse the vibration and buckling behaviour of FG sandwich beams. The investigation into free vibration behaviour of FG beams is also carried out by mesh-free techniques in [17-19] and homogenization approach in [20].

One of the key parameters in the analysis of the vibration behaviour of structures is the boundary conditions. For the real structures, the imperfect boundary conditions known as non-classical conditions consisting of rotational and translational displacements are essential and need to be considered. There are many publications relating to the investigation of free vibration under imperfect support, which mostly focus on isotropic beams such as Hsu et al. [21] with Adomian modified decomposition method using FOBT; Sari [22, 23] with Chebyshev collocation method based on CBT and FOBT. Regarding the FG beams, Simsek and Cansiz [24] analysed dynamic behaviour of elastically connected double-FG beam with elastic supports under moving harmonic load. Shahba et al. [25] studied free vibration response of Timoshenko axially FG beams. Wattanasakulpong and Mao [26] also applied Chebyskev collocation to consider the free vibration of FG beams using FOBT. Although several studies focused on vibration analysis of FG beams and FG sandwich beams can be found, as far as the knowledge of the authors, there has been no study based on the state space approach. This approach was only applied by Khdeir and Reddy [27-29] to study static and dynamic behaviour of cross-ply laminated beams under arbitrary boundary conditions.

In the present work, the state space approach is applied to analyse free vibration behaviour of FG sandwich beams with various theories (CBT, FOBT and HOBT) and different boundary conditions. Governing equations of motions are derived using Hamilton's principle. Nine combinations classical boundary conditions created from clamped, hinged, pinned and free and three combinations of non-classical boundary conditions created from elastic support with translational and rotational springs are considered. The effect of material parameter, arrangement of layers and slenderness ratio on fundamental frequencies of FG sandwich beams is examined.

2. Theoretical formulation

2.1. FG sandwich beams

Consider a FG sandwich beam with the span and cross-section being L and $b \times h$, respectively, as shown in Fig.1a. It is assumed that the beams are manufactured by the mixtures of isotropic metal and ceramic, in which the volume fraction of the constituents varies through the beam depth. In this paper, FG beams (Type A) and two types of FG sandwich beams namely FG-faces ceramic-core (Type B) and FG-core homogeneous-faces (Type C) are investigated.

2.1.1. Type A: FG beams

The beam is made of metal-ceramic material (Fig.1b) with the volume fraction of ceramic V_c given by:

$$V_c(z) = \left(\frac{z}{h} + \frac{1}{2}\right)^p \tag{1}$$

where p is the material parameter.

2.1.2. Type B: sandwich beams with FG skins and ceramic core

The ceramic core is inserted between two FG sheets in which the rich ceramic ones are in contact with the core (Fig.1c). The volume fractions of ceramic in layer n^{th} $(n = \overline{1-3})$ is:

$$\begin{cases} V_c^{(1)}(z) = \left(\frac{z-h_0}{h_1-h_0}\right)^p & for\ z \in [h_0, h_1] \\ V_c^{(2)}(z) = 1 & for\ z \in [h_1, h_2] \\ V_c^{(3)}(z) = \left(\frac{z-h_3}{h_2-h_3}\right)^p & for\ z \in [h_2, h_3] \end{cases} \tag{2}$$

2.1.3. Type C: sandwich beams with FG core and homogeneous skins

In these beams, metal and ceramic are placed on the metal-rich and ceramic-rich faces of the FG core (Fig.1d), respectively. By this way, delamination which occurs in conventional composites can be avoided. The volume fractions of ceramic $V_c^{(n)}$ in layer n^{th} is:

$$\begin{cases} V_c^{(1)}(z) = 0 & for\ z \in [h_0, h_1] \\ V_c^{(2)}(z) = \left(\frac{z-h_1}{h_2-h_1}\right)^p & for\ z \in [h_1, h_2] \\ V_c^{(3)}(z) = 1 & for\ z \in [h_2, h_3] \end{cases} \tag{3}$$

The material properties including Young's modulus $E(z)$, Poisson's ratio $\upsilon(z)$ and mass density $\rho(z)$ are expressed in the general form:

$$P(z) = (P_c - P_m)V_c(z) + P_m \tag{4}$$

where the subscripts c and m represent ceramic and metal while $P(z)$ represents $E(z), \upsilon(z)$ and $\rho(z)$, respectively. It should be noted that Poisson's ratio $\upsilon(z)$ is assumed to be constant in this paper. Figs. (2a), (2b) and (2c) present sample results of the variations of the Young's modulus $E(z)$ through the thickness for Type A, Type B and Type C, respectively.

2.2. Kinematics

Assuming that the deformation of FG sandwich beam is only in $x - z$ plane and let $u(x, z, t)$ and $w(x, z, t)$ be the axial and transverse displacement components at an arbitrary point (x, z). These components can be expressed as shown in Eq. (5):

$$u(x, z, t) = U + z(\psi_0 w' + \psi_1 \varphi) + \psi_3(-zw' + f\varphi) \tag{5a}$$

$$w(x, z, t) = W \tag{5b}$$

where $U = U(x, t)$ and $W = W(x, t)$ represent the displacement components of a point on the beam's neutral axis along x and z directions while $\varphi = \varphi(x, t)$ is the rotational angle of the cross-section about y-axis compared to the undeform position. The shape function considered in this paper is $f = z - \frac{4z^2}{3h^3}$

which is proposed by Reddy [30]. For brevity, prime (‘) represents the partial differentiation of the quantities to x. $\psi_0 = -1, \psi_1 = 1$ and $\psi_3 = 1$ are the constants defining the axial displacements over the thickness, which can be used to determine the shear deformation theory considered. ψ_0, ψ_1 and ψ_3 only exist in the formulations for the CBT, FOBT and HOBT, respectively. By using these constants, one can describe the CBT, FOBT and HOBT formulations simultaneously.

The strains related to the displacement field in Eq. (5) are:

$$\varepsilon_x = \frac{\partial u}{\partial x} = U' + z[(\psi_0 - \psi_3)W'' + \psi_1\varphi'] + \psi_3 f\varphi' \tag{6a}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = (1 + \psi_0 - \psi_3)W' + \psi_1\varphi \tag{6b}$$

2.3. Variational formulation

In order to derive the governing equations of motions and boundary conditions, Hamilton’s principle is employed:

$$\int_{t_1}^{t_2} (\delta\Pi - \delta K)\, dt = 0 \tag{7}$$

where $\delta\Pi$ and δK denote the virtual variation of the strain energy and kinetic energy.

The virtual variation of the strain energy is given by:

$$\delta\Pi = \int_0^L \int_0^b \left[\sum_{n=1}^{3} \int_{h_{n-1}}^{h_n} (\sigma_x \delta\varepsilon_x + \sigma_{xz}\delta\gamma_{xz})\right] dzdydx$$

$$= \int_0^T \int_0^L N_x \delta U' + M_x[(\psi_0 - \psi_3)\delta W'' + \psi_1\delta\varphi'] + P_x\psi_3\delta\varphi' + Q_{xz}[(1 + \psi_0 - \psi_3)\delta w' + \psi_1\delta\varphi] dxdt \tag{8}$$

where the stress resultants N_x, M_x, P_x and Q_{xz} can be defined as:

$$N_x = \sum_{n=1}^{3} \int_{h_{n-1}}^{h_n} \sigma_x^{(n)} bdz \tag{9a}$$

$$M_x = \sum_{n=1}^{3} \int_{h_{n-1}}^{h_n} \sigma_x^{(n)} bzdz \tag{9b}$$

$$P_x = \sum_{n=1}^{3} \int_{h_{n-1}}^{h_n} \sigma_x^{(n)} bfdz \tag{9c}$$

$$Q_{xz} = \sum_{n=1}^{3} \int_{h_{n-1}}^{h_n} \sigma_{xz}^{(n)} bdz. \tag{9d}$$

The virtual variation of the kinetic energy can be determined as:

$$\delta K = \int_0^T \int_0^L (\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w})\rho(z)dxdt$$

$$= \int_0^T \int_0^L \{ I_0\big(\dot{U}\delta\dot{U} + \dot{W}\delta\dot{W}\big) + I_1\big[\psi_0\big(\dot{W}'\delta\dot{U} + \dot{U}\delta\dot{W}'\big) + \psi_1\big(\dot{\varphi}_x\delta\dot{U} + \dot{U}\delta\dot{\varphi}_x\big) - \psi_3\big(\dot{W}'\delta\dot{U} +$$

$$U\delta W'+I2-\psi 0W'\delta W'+\psi 1\varphi x\delta\varphi x+\psi 3W'\delta W'+If\psi 3\varphi x\delta U+U\delta\varphi x-Izf\psi 3\varphi x\delta W'+W'\delta\varphi x+If2\psi 3\varphi$$

$$x\delta\varphi xdxdt \tag{10}$$

where

$$\big(I_0, I_1, I_2, I_f, I_{fz}, I_{f^2}\big) = \sum_{n=1}^{3} \int_{h_{n-1}}^{h_n} (1, z, z^2, f, zf, f^2)\rho(z)dz \tag{11}$$

2.3. Constitutive equations

The stress in layer n[th] is given as:

$$\sigma_x^{(n)} = E^{(n)}\varepsilon_x \tag{12a}$$

$$\sigma_{xz}^{(n)} = \frac{E^{(n)}}{2\left[1+\nu^{(n)}\right]}\gamma_{xz} \tag{12b}$$

By substituting Eq. (6) into Eq. (12), the stress resultants are obtained as:

$$\begin{Bmatrix} N_x \\ M_x \\ P_x \\ Q_{xz} \end{Bmatrix} = \begin{bmatrix} A & B & C & 0 \\ B & D & F & 0 \\ C & F & H & 0 \\ 0 & 0 & 0 & A_s \end{bmatrix} \begin{Bmatrix} U^{'} \\ (\psi_0 - \psi_3)W'' + \psi_1\varphi' \\ \psi_3\varphi' \\ (1 + \psi_0 - \psi_3)\delta W' + \psi_1\delta\varphi \end{Bmatrix} \tag{13}$$

where

$$(A, B, D, C, F, H) = \int_A \; E(1, z, z^2, f, zf, f^2)\, dA \tag{14a}$$

$$A_S = \int_A \; \frac{E}{2(1+\upsilon)} dA. \tag{14b}$$

2.4. Governing equations of motion

By substituting Eqs. (8) and (10) into Eq. (7), integrating the equation by part, and collecting the coefficients of δU, $\delta\varphi$ and δW, the governing equations of motion are obtained as:

$$N'_x = I_0\ddot{U} + I_1\big(\psi_0\ddot{W}' + \psi_1\ddot{\varphi} - \psi_3\ddot{W}'\big) + \psi_3 I_f\ddot{\varphi} \tag{15a}$$

$$M''_x(\psi_3 - \psi_0) = I_0\ddot{W} - I_1\left(\psi_0\ddot{U}' - \psi_3\ddot{U}'\right) + I_2\left(\psi_0\ddot{W}'' - \psi_3\ddot{W}''\right) + \psi_3 I_{zf}\ddot{\varphi}' \tag{15b}$$

$$M'_x\psi_1 + P'_x\psi_3 - Q_{xz}(\psi_1 + \psi_3 f') = I_1\psi_1\ddot{U} + I_2\psi_1\ddot{\varphi} + I_f\psi_3\ddot{U} - I_{zf}\psi_3\ddot{W}' + I_{f^2}\psi_3\ddot{\varphi} \tag{15c}$$

By using the state space approach [27-29], the displacement components can be expressed as:

$$\begin{Bmatrix} U(x,t) \\ \varphi(x,t) \\ W(x,t) \end{Bmatrix} = \begin{bmatrix} U(x) \\ \varphi(x) \\ W(x) \end{bmatrix} e^{i\omega t} \tag{16}$$

where ω is the eigen-frequency.

By substituting Eq. (16) into Eq. (15), a system of ordinary differential equations is obtained:

For CBT:

$$U'' = e_1 U + e_2 W' + e_3 W''' \tag{17a}$$

$$W^{iv} = e_4 U' + e_5 W + e_6 W''. \tag{17b}$$

For FOBT:

$$U'' = c_1 U + c_2\varphi + c_3 W' \tag{18a}$$

$$\varphi'' = c_4 U + c_5\varphi + c_6 W' \tag{18b}$$

$$W'' = c_7\varphi' + c_8 W. \tag{18c}$$

For HOBT:

$$U'' = b_1 U + b_2\varphi + b_3 W' + b_4 W''' \tag{19a}$$

$$\varphi'' = b_5 U + b_6\varphi + b_7 W' + b_8 W''' \tag{19b}$$

$$W^{iv} = b_9 U' + b_{10}\varphi' + b_{11} W + b_{12} W'' \tag{19c}$$

where the e_n, c_n and b_n are constants coefficients and are described in the Appendix.

The systems of Eqs. (17-19) can be converted into a matrix form as:

$$Z'(x) = TZ(x) \tag{20}$$

where

$Z(x) = \{U, U', W, W', W'', W'''\}$ for CBT

$Z(x) = \{U, U', \varphi, \varphi', W, W'\}$ for FOBT

$Z(x) = \{U, U', \varphi, \varphi', W, W', W'', W'''\}$ for HOBT

and matrix T can be seen in the Appendix.

A formal solution of Eq. (20) is given by:

$$Z(x) = e^{Tx} K \tag{21}$$

where K is a constant column vector determined from the boundary conditions at $x = \pm L/2$; and e^{Tx} is the general matrix solution of Eq. (20) which is given as:

$$e^{Tx} = [E] \begin{bmatrix} e^{\lambda_1 x} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_6 x} \end{bmatrix} [E]^{-1} \tag{22}$$

where $\lambda_i (i = \overline{1,6})$ for CBT and FOBT, $\lambda_i (i = \overline{1,8})$ for HOBT and $[E]$ are eigenvalues and corresponding matrix of eigenvectors, respectively, associated with the matrix T.

2.5. Boundary conditions (BCs)

In this paper, both classical and non-classical boundary conditions (see Fig. 3 and Table 1) are investigated. Translational and rotational springs are considered to model non-classical boundary conditions. The boundary conditions are shown in Fig. 3 and described in Table 1.

2.5.1. Classical BCs

Boundary conditions can be expressed in terms of unknown function $Z(x)$ as follows:

For CBT:

Clamped (C): $$U = W = W' = 0 \tag{23a}$$

Hinged (H): $$U = W = BU' - DW'' = 0 \tag{23b}$$

Pinned (P): $AU' - BW'' = W = BU' - DW'' = 0$ (23c)

Free (F): $AU' - BW'' = BU' - DW'' = BU'' - DW''' + I_1\omega^2 U - I_2\omega^2 W' = 0$ (23d)

For FOBT:

Clamped (C): $U = \varphi = W = 0$ (24a)

Hinged (H): $U = W = BU' + D\varphi' = 0$ (24b)

Pinned (P): $AU' + B\varphi' = W = BU' + D\varphi' = 0$ (24c)

Free (F): $AU' + B\varphi' = BU' + D\varphi' = A_s(\varphi + W') = 0$ (24d)

For HOBT:

Clamped (C): $U = \varphi = W = W' = 0$ (25a)

Hinged (H): $U = W = AU' - FW'' + H\varphi' = BU' - DW'' + F\varphi' = 0$ (25b)

Pinned (P): $AU' - BW'' + C\varphi' = W = AU' - FW'' + H\varphi' = BU' - DW'' + F\varphi' = 0$(25c)

Free (F): $AU' - BW'' + C\varphi' = BU' - DW'' + F\varphi' = BU'' - DW''' + F\varphi'' + I_1\omega^2 U +$

$J_2\omega^2\varphi - I_2\omega^2 W' = AU' - FW'' + H\varphi' = 0$ (25d)

2.5.2. Non – classical BCs

In the company of clamped and hinged conditions in section 2.5.1, elastic supported boundary, which is used to consider imperfect supports, can be described as:

For CBT:

$$U = BU' - DW'' \pm k_R W' = BU'' - DW''' + I_1\omega^2 U - I_2\omega^2 W' \pm k_T W = 0 \quad (26)$$

For FOBT:

$$U = A_s(\varphi + W') \pm k_T W = BU' + D\varphi' \pm k_R\varphi = 0 \quad (27)$$

For HOBT:

$$U = BU'' - DW''' + F\varphi'' + I_1\omega^2 U + J_2\omega^2\varphi - I_2\omega^2 W' \pm k_T W = BU' - DW'' + F\varphi' \pm k_R\varphi = AU' - FW'' + H\varphi' \pm k_R W' = 0 \quad (28)$$

where k_T and k_R are the translational and rotational spring stiffness; '+' for the left end and '-' for the right end. For simplicity, the dimensionless spring factors for translational and rotational support are defined as $\beta_T = \frac{k_T L}{E_m h}$; $\beta_R = \frac{12 k_R L}{E_m h^3}$, respectively.

Substituting Eq. (21) into Eqs. (23-28), a homogeneous system of equations is obtained as:

$$G_{ij}K_j = 0, \quad (i,j) = \overline{1,6} \text{ for CBT, FOBT and } (i,j) = \overline{1,8} \text{ for HOBT} \quad (29)$$

where

$$[G(x)] = [E]\begin{bmatrix} e^{\lambda_1 x} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_6 x} \end{bmatrix}[E]^{-1} \quad (30)$$

By setting the determinant of G_{ij} to zero, the natural frequency ω can be determined. It should be noted that a trial and error procedure need to be used to obtain the natural frequency values due to the attendant of unknown ω in matrix T.

3. Results and discussion

In this section, the state space approach is applied to investigate the fundamental frequencies of FG beams and FG sandwich beams using various theories (CBT, FOBT and HOBT). Nine combinations of classical BCs created from Clamped (C), Hinged (H), Pinned (P) and Free (F) and three combinations of non-classical BCs created from Elastic Support (E) with translational and rotational springs are considered. The base materials of these beams are ceramic (Al_2O_3) and metal (Al) with material properties of E_c =380 GPa, υ_c =0.3, ρ_c =3960 kg/m^3 and E_m=70 GPa, υ_m =0.3, ρ_m =2702 kg/m^3, respectively. The material parameter p varies from 0 to 10 while the slenderness ratios L/h are 5 and 20 representing thick and thin beams. For simplicity, the non-dimensional natural frequency is defined as $\omega = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$ for classical BCs and $\overline{\omega} = \omega L\sqrt{\frac{\rho_m}{E_m}}$ for non-classical BCs.

3.1. Classical boundary conditions:

For verification purpose, Tables 2-7 show the non-dimensional natural frequency of FG beams (Type A) and FG sandwich beams (Type B and C) with various BCs. The material parameters are 0, 1 and 10. All the results relating to thin and thick beams under C-C, P-P and C-F boundary conditions agree well with Simsek's [13] for FG beams, Vo et al. [15] and Nguyen et al. [16] for FG sandwich beams. Furthermore, for thin beams (L/h=20), the present results are in line with Jin and Wang's [3], which applied Quadrature Element Method using CBT, under nine classical BCs. It can be seen that natural frequencies reduce significantly along with the increase of material parameter p. Regarding the relation between frequencies and boundary conditions, the downtrend of natural frequencies for these beams is $\omega_{F-F} > \omega_{C-C} > \omega_{H-C} > \omega_{P-C} > \omega_{P-F} > \omega_{H-F} > \omega_{H-H} > \omega_{P-P} > \omega_{C-F}$. It is also worth noting that, $\omega_{C-C} > \omega_{F-F}$ for CBT but $\omega_{C-C} < \omega_{F-F}$ when the FOBT and HOBT are considered. In addition, the natural frequencies under which combined of H and P for $p = 0$ (isotropic material) are the same but for FG beams and FG sandwich beams, there is a slight difference. For FG beams, the difference in the natural frequencies due to shear deformation effect is minor for all BCs with a maximum of 1.8% (Table 3). However, a large difference can be seen between the results from CBT and FOBT/HOBT for FG sandwich thick beams: the highest errors are observed for C-C (11- 22%); H-C and P-C (8-12%); F-F, P-F, H-F, H-H, P-P (2-5%) and the lowest errors are for C-F (1-2%) (Tables 5 and 7).

3.2. Non – classical boundary conditions:

In order to verify present theory further, Tables 8 and 9 present the fundamental frequency of FG beams (L/h=10) with elastic support at both ends (E-E), C-E and H-E with various material parameters while the spring stiffness varies from 10^{-4} to 10^{6} ($\beta_T = \beta_R$). It can be seen that the results of FG beams with FOBT have excellent agreement with Wattanasakulpong and Mao [26], while the HOBT provides slightly higher values in comparison with FOBT. It is seen that the frequencies increase with an increase in the spring stiffness, and similar to the case of classical BCs, they decrease consistent with the values of material parameters. It is also seen that the frequency for C-E boundary condition is highest, followed by E-E and H-E conditions. Tables 10 and 11 provide the frequency of non-rotational and non-translational FG sandwich beams, respectively. For the non-rotational beams, the rotational spring stiffness is set as $\beta_R = 10^6$ (considered as clamped), whereas the translational spring factor β_T varies from 10^{-4} to 10^6. Similarly, $\beta_T = 10^6$ (considered as supported) and $\beta_R = 10^{-4} \rightarrow 10^6$ are adopted for non-translational case. The trend of variation of the frequencies in these cases is comparable to the response of FG beams (L/h=10, p=0.5) in Figs. 4 and 5. It is seen that for the non-rotational beams, the frequencies increase rapidly when β_T changes from 10^{-4} to 10, and plateau for higher values of β_T; however, for the non-translational ones, they only change for the interval $\beta_T \in [10^{-1};\ 10^3]$ or $[10;\ 10^2]$ using FOBT or HOBT. In addition, the variation of translational spring stiffness results in a larger range of the frequencies compared to the change of rotational spring stiffness. This tendency can be seen more clearly in Fig. 6, which presents the frequencies of C-E FG beams (L/h=10, p=1) according to various rotational and translational spring factors (both β_T and β_R range from 10^{-4} to 10^6). It can be seen that the frequencies increase slightly in accordance with rotational stiffness and sharply in line with translational spring factors.

4. Conclusions

In this paper, state space approach is applied to analyse free vibration behaviour of FG sandwich beams with various theories (CBT, FOBT and HOBT). Hamilton's principle is applied to derive the governing equations of motion and boundary conditions. Numerical results for the free vibration behaviour of FG and FG sandwich beams under classical and non-classical boundary conditions are investigated and show good agreement with those from the literature. The data also expose that the influences of hinged and pinned conditions to the natural frequencies are significantly different for FG sandwich beams. Regarding the imperfect boundary conditions, the numerical results reveal that the frequencies increase with an increase of spring factors and the effect of translational stiffness the fundamental frequency is more pronounced compared with the rotational stiffness.

APPENDIX

1. The coefficients in Eqs. (17)-(19).

For Eq. (17):

$$e_1 = \frac{-I_0\omega^2}{A},\ e_2 = \frac{I_1\omega^2}{A},\ e_3 = \frac{B}{A},\ e_4 = \frac{-\left(Be_1 + I_1\omega^2\right)}{(Be_3 - D)},\ e_5 = \frac{-I_0\omega^2}{(Be_3 - D)},\ e_6 = \frac{\left(I_2\omega^2 - Be_2\right)}{(Be_3 - D)}\ .$$

For Eq. (18):

$$c_1 = \frac{(e_{2a}e_3 - e_2e_{3a})}{C_0},\ c_2 = \frac{(e_{2a}e_4 - e_2e_{4a})}{C_0},\quad c_3 = \frac{-e_2e_{5a}}{C_0},$$

$$c_4 = \frac{(e_1e_{3a} - e_{1a}e_3)}{C_0},\ c_5 = \frac{(e_1e_{4a} - e_{1a}e_4)}{C_0},\quad c_6 = \frac{e_1e_{5a}}{C_0},$$

$$c_7 = \frac{-A_s}{A_s} = -1,\ c_8 = \frac{-I_0\omega^2}{A_s},$$

$$e_1 = A,\ e_2 = B,\ e_3 = -I_0\omega^2,\ e_4 = -I_1\omega^2,$$

$$e_{1a} = e_2,\ e_{2a} = D,\ e_{3a} = e_4,\ e_{4a} = (-I_2\omega^2 + A_s),$$

$$e_{5a} = A_s,$$

$$C_0 = e_1e_{2a} - e_{1a}e_2.$$

For Eq. (19)

$$b_1 = \frac{e_1e_8 - e_3e_4}{e_2e_8 - e_4^2}, b_2 = \frac{e_3e_8 - e_4e_7}{e_2e_8 - e_4^2}, b_3 = \frac{e_5e_8 - e_4e_9}{e_2e_8 - e_4^2}, b_2 = \frac{-e_6e_8 + e_4e_{10}}{e_2e_8 - e_4^2},$$

$$b_5 = \frac{e_2e_3 - e_1e_4}{e_2e_8 - e_4^2}, b_6 = \frac{e_2e_7 - e_3e_4}{e_2e_8 - e_4^2}, b_7 = \frac{e_2e_9 - e_4e_5}{e_2e_8 - e_4^2}, b_8 = \frac{-e_2e_{10} + e_4e_6}{e_2e_8 - e_4^2},$$

$$e_1 = -I_0\omega^2;\ e_2 = A;\ e_3 = -J_1\omega^2;\ e_4 = C;\ e_5 = I_1\omega^2;\ e_6 = -B;$$

$$e_7 = K_2\omega^2 + A_s;\ e_8 = H;\ e_9 = -J_2\omega^2;\ e_{10} = -F.$$

2. Matrix T in Eq.(20):

For CBT:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ e_1 & 0 & 0 & e_2 & 0 & e_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & e_4 & e_5 & 0 & e_6 & 0 \end{bmatrix}$$

For FOBT:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ c_1 & 0 & c_2 & 0 & 0 & c_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ c_4 & 0 & c_5 & 0 & 0 & c_6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & c_7 & c_8 & 0 \end{bmatrix}$$

For HOBT:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 0 & b_2 & 0 & 0 & b_3 & 0 & b_4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ b_5 & 0 & b_6 & 0 & 0 & b_7 & 0 & b_8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & b_9 & 0 & b_{10} & b_{11} & 0 & b_{12} & 0 \end{bmatrix}$$

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Table 1: Boundary conditions (BCs)

Type	Natural BCs	Essential BCs
Classical	N_x	U
	$M_x\psi_1 + P_x\psi_3$	φ
	$M_x(\psi_3 - \psi_0)$	W'
	$(\psi_0 - \psi_3)M'_x - (1 + \psi_0 - \psi_3)Q_{xz}$	W
Non-classical	Rotational support	$\begin{cases} M_x\psi_1 + P_x\psi_3 \pm k_R\varphi \\ M_x(\psi_3 - \psi_0) \pm k_R W' \end{cases}$
	Translational support	$(\psi_0 - \psi_3)M'_x - (1 + \psi_0 - \psi_3)Q_{xz} \pm k_R W$

Table 2: Non-dimensional fundamental frequency of FG beams (Type A) with various BCs

(L/h=20, $\bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$)

p	Theory		BCs								
			C–C	F–F	S–C	P–C	P–F	S–F	S–S	P–P	C–F
0	CBT	Present	12.4143	12.3666	8.5559	8.5559	8.5420	8.5420	5.4778	5.4778	1.9528
		[3]	12.4143	12.3666	8.5558	8.5558	8.5419	8.5419	5.4777	5.4777	1.9525
		[13]	12.4142	-	-	-	-	-	-	5.4777	1.9525
	FOBT	Present	12.2205	12.3184	8.4818	8.4818	8.5107	8.5107	5.4604	5.4604	1.9499
		[15]	12.2202	-	-	-	-	-	-	5.4603	1.9496
		[13]	12.2235	-	-	-	-	-	-	5.4603	1.9496
	HOBT	Present	12.2222	12.3187	8.4818	8.4818	8.5104	8.5104	5.4604	5.4604	1.9496
		[15]	12.2228	-	-	-	-	-	-	5.4603	1.9496
		[13]	12.2238	-	-	-	-	-	-	5.4603	1.9495
1	CBT	Present	9.5554	9.5162	6.6682	6.5857	6.5747	6.5747	4.2168	4.2168	1.5030
		[3]	9.5554	9.5160	6.6677	6.5855	6.5743	6.5691	4.5161	4.2163	1.5029
		[13]	9.5554	-	-	-	-	-	-	4.2163	1.5029
	FOBT	Present	9.4297	9.4853	6.6185	6.5375	6.5544	6.5489	4.2039	4.2054	1.5010
		[15]	9.4311	-	-	-	-	-	-	4.2039	1.5011
		[13]	9.4314	-	-	-	-	-	-	4.2051	1.5010
	HOBT	Present	9.4304	9.4851	6.6187	6.5377	6.5541	6.5489	4.5024	4.2051	1.5013
		[15]	9.4328	-	-	-	-	-	-	4.2039	1.5011
		[13]	9.4316	-	-	-	-	-	-	4.2050	1.5011
10	CBT	Present	8.0559	8.0196	5.6070	5.5523	5.5419	5.5419	3.5548	3.5548	1.2674
		[3]	8.0556	8.0194	5.6069	5.5519	5.5416	5.5365	3.7568	3.5547	1.2671
		[13]	8.0556	-	-	-	-	-	-	3.5547	1.2671
	FOBT	Present	7.9103	7.9838	5.5503	5.4962	5.5180	5.5131	3.5404	3.5418	1.2649
		[15]	7.9128	-	-	-	-	-	-	3.5405	1.2650
		[13]	7.9128	-	-	-	-	-	-	3.5416	1.2650
	HOBT	Present	7.8844	7.9764	5.5394	5.4857	5.5136	5.5086	3.7387	3.5391	1.2647
		[15]	7.8862	-	-	-	-	-	-	3.5379	1.2645
		[13]	7.8859	-	-	-	-	-	-	3.5390	1.2645

Table 3: Non-dimensional fundamental frequency of FG beams (Type A) with various BCs

(L/h=5, $\bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$)

p	Theory		BCs								
			C–C	F–F	S–C	P–C	P–F	S–F	S–S	P–P	C–F
0	CBT	Present	12.4143	12.3666	8.5559	8.5559	8.5420	8.5420	5.4778	5.4778	1.9528
		[3]	12.4143	12.3666	8.5558	8.5558	8.5419	8.5419	5.4777	5.4777	1.9525
		[13]	12.4142	-	-	-	-	-	-	5.4777	1.9525
	FOBT	Present	12.2205	12.3184	8.4818	8.4818	8.5107	8.5107	5.4604	5.4604	1.9499
		[15]	12.2202	-	-	-	-	-	-	5.4603	1.9496
		[13]	12.2235	-	-	-	-	-	-	5.4603	1.9496
	HOBT	Present	12.2222	12.3187	8.4818	8.4818	8.5104	8.5104	5.4604	5.4604	1.9496
		[15]	12.2228	-	-	-	-	-	-	5.4603	1.9496
		[13]	12.2238	-	-	-	-	-	-	5.4603	1.9495
1	CBT	Present	9.5554	9.5162	6.6682	6.5857	6.5747	6.5747	4.2168	4.2168	1.5030
		[3]	9.5554	9.5160	6.6677	6.5855	6.5743	6.5691	4.5161	4.2163	1.5029
		[13]	9.5554	-	-	-	-	-	-	4.2163	1.5029
	FOBT	Present	9.4297	9.4853	6.6185	6.5375	6.5544	6.5489	4.2039	4.2054	1.5010
		[15]	9.4311	-	-	-	-	-	-	4.2039	1.5011
		[13]	9.4314	-	-	-	-	-	-	4.2051	1.5010
	HOBT	Present	9.4304	9.4851	6.6187	6.5377	6.5541	6.5489	4.5024	4.2051	1.5013
		[15]	9.4328	-	-	-	-	-	-	4.2039	1.5011
		[13]	9.4316	-	-	-	-	-	-	4.2050	1.5011
10	CBT	Present	8.0559	8.0196	5.6070	5.5523	5.5419	5.5419	3.5548	3.5548	1.2674
		[3]	8.0556	8.0194	5.6069	5.5519	5.5416	5.5365	3.7568	3.5547	1.2671
		[13]	8.0556	-	-	-	-	-	-	3.5547	1.2671
	FOBT	Present	7.9103	7.9838	5.5503	5.4962	5.5180	5.5131	3.5404	3.5418	1.2649
		[15]	7.9128	-	-	-	-	-	-	3.5405	1.2650
		[13]	7.9128	-	-	-	-	-	-	3.5416	1.2650
	HOBT	Present	7.8844	7.9764	5.5394	5.4857	5.5136	5.5086	3.7387	3.5391	1.2647
		[15]	7.8862	-	-	-	-	-	-	3.5379	1.2645
		[13]	7.8859	-	-	-	-	-	-	3.5390	1.2645

Table 4: Non-dimensional fundamental frequency of FG sandwich beams with ceramic core (Type B) for various BCs (L/h=20, $\bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$)

Scheme	p	Theory	BCs								
			C–C	F–F	S–C	P–C	P–F	S–F	S–S	P–P	C–F
1-1-1	0	CBT	12.4145	12.3670	8.5560	8.5560	8.5420	8.5420	5.4780	5.4780	1.9525
		FOBT	12.2205	12.3185	8.4815	8.4815	8.5105	8.5105	5.4605	5.4605	1.9500
		HOBT	12.2225	12.3185	8.4820	8.4820	8.5105	8.5105	5.4605	5.4605	1.9500
		HOBT [15]	12.2228	-	-	-	-	-	-	5.4603	1.9496
	1	CBT	9.1575	9.1260	6.3110	6.3110	6.3020	6.3020	4.0405	4.0405	1.4405
		FOBT	9.0610	9.1020	6.2745	6.2745	6.2865	6.2865	4.0320	4.0320	1.4390
		HOBT	9.0720	9.1045	6.2785	6.2785	6.2880	6.2880	4.0330	4.0330	1.4390
		HOBT [15]	9.0722	-	-	-	-	-	-	4.0328	1.4388
	10	CBT	6.7305	6.7080	4.6385	4.6385	4.6320	4.6320	2.9700	2.9700	1.0585
		FOBT	6.6805	6.6955	4.6195	4.6195	4.6240	4.6240	2.9655	2.9655	1.0580
		HOBT	6.6925	6.6985	4.6240	4.6240	4.6260	4.6260	2.9665	2.9665	1.0580
		HOBT [15]	6.6924	-	-	-	-	-	-	2.9662	1.0578
1-2-1	0	CBT	12.4145	12.3670	8.5560	8.5560	8.5420	8.5420	5.4780	5.4780	1.9525
		FOBT	12.2205	12.3185	8.4815	8.4815	8.5105	8.5105	5.4605	5.4605	1.9500
		HOBT	12.2225	12.3185	8.4820	8.4820	8.5105	8.5105	5.4605	5.4605	1.9500
		HOBT [15]	12.2228	-	-	-	-	-	-	5.4603	1.9496
	1	CBT	9.7410	9.7070	6.7135	6.7135	6.7035	6.7035	4.2980	4.2980	1.5320
		FOBT	9.6315	9.6800	6.6715	6.6715	6.6860	6.6860	4.2885	4.2885	1.5305
		HOBT	9.6410	9.6820	6.6750	6.6750	6.6870	6.6870	4.2890	4.2890	1.5305
		HOBT [15]	9.6411	-	-	-	-	-	-	4.2889	1.5304
	10	CBT	7.5815	7.5565	5.2250	5.2250	5.2180	5.2180	3.3455	3.3455	1.1925
		FOBT	7.5210	7.5415	5.2020	5.2020	5.2080	5.2080	3.3400	3.3400	1.1915
		HOBT	7.5310	7.5440	5.2060	5.2060	5.2095	5.2095	3.3410	3.3410	1.1915
		HOBT [15]	7.5311	-	-	-	-	-	-	3.3406	1.1915
2-2-1	0	CBT	12.4145	12.3670	8.5560	8.5560	8.5420	8.5420	5.4780	5.4780	1.9525
		FOBT	12.2205	12.3185	8.4815	8.4815	8.5105	8.5105	5.4605	5.4605	1.9500
		HOBT	12.2225	12.3185	8.4820	8.4820	8.5105	8.5105	5.4605	5.4605	1.9500
		HOBT [15]	12.2228	-	-	-	-	-	-	5.4603	1.9496
	1	CBT	9.4480	9.4150	6.5195	6.5115	6.5020	6.5020	4.1690	4.1690	1.4860
		FOBT	9.3445	9.3895	6.4800	6.4720	6.4850	6.4850	4.1595	4.1595	1.4845
		HOBT	9.3545	9.3915	6.4835	6.4755	6.4865	6.4865	4.1910	4.1605	1.4845
		HOBT [15]	9.3550	-	-	-	-	-	-	4.1602	1.4844
	10	CBT	7.1750	7.1500	4.9740	4.9450	4.9375	4.9375	3.1660	3.1660	1.1285
		FOBT	7.1185	7.1360	4.9520	4.9235	4.9285	4.9275	3.1605	3.1610	1.1275
		HOBT	7.1290	7.1385	4.9560	4.9275	4.9300	4.9290	3.2695	3.1620	1.1280
		HOBT [15]	7.1296	-	-	-	-	-	-	3.1613	1.1276

Table 5: Non-dimensional fundamental frequency of FG sandwich beams with ceramic core (Type B) for various BCs (L/h=5, $\bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$)

Scheme	p	Theory	BCs								
			C–C	F–F	S–C	P–C	P–F	S–F	S–S	P–P	C–F
1-1-1	0	CBT	12.1827	11.5140	8.4062	8.4062	8.2012	8.2012	5.3955	5.3955	1.9385
		FOBT	9.9977	11.0000	7.4652	7.4652	7.8070	7.8070	5.1525	5.1525	1.8945
		HOBT	10.0670	11.0020	7.4872	7.4872	7.8080	7.8080	5.1528	5.1528	1.8953
		HOBT [15]	10.0678	-	-	-	-	-	-	5.1528	1.8952
	1	CBT	9.0040	8.5565	6.2120	6.2120	6.0755	6.0755	3.9860	3.9860	1.4310
		FOBT	7.8235	8.2892	5.7197	5.7197	5.8735	5.8735	3.8628	3.8628	1.4090
		HOBT	7.9572	8.3162	5.7755	5.7755	5.8942	5.8942	3.8758	3.8758	1.4115
		HOBT [15]	7.9580	-	-	-	-	-	-	3.8755	1.4115
	10	CBT	6.6205	6.2995	4.5675	4.5675	4.4698	4.4698	2.9308	2.9308	1.0520
		FOBT	5.9755	6.1590	4.3043	4.3043	4.3635	4.3635	2.8660	2.8660	1.0405
		HOBT	6.1237	6.1907	4.3658	4.3658	4.3878	4.3878	2.8810	2.8810	1.0433
		HOBT [15]	6.1240	-	-	-	-	-	-	2.8808	1.0431
1-2-1	0	CBT	12.1827	11.5140	8.4062	8.4062	8.2012	8.2012	5.3955	5.3955	1.9385
		FOBT	9.9977	11.0000	7.4652	7.4652	7.8070	7.8070	5.1525	5.1525	1.8945
		HOBT	10.0670	11.0020	7.4872	7.4872	7.8080	7.8080	5.1528	5.1528	1.8953
		HOBT [15]	10.0678	-	-	-	-	-	-	5.1528	1.8952
	1	CBT	9.5762	9.0962	6.6070	6.6070	6.4605	6.4605	4.2395	4.2395	1.5220
		FOBT	8.2545	8.7952	6.0532	6.0532	6.2325	6.2325	4.1003	4.1003	1.4973
		HOBT	8.3697	8.8165	6.1005	6.1005	6.2490	6.2490	4.1108	4.1108	1.4993
		HOBT [15]	8.3705	-	-	-	-	-	-	4.1105	1.4992
	10	CBT	7.4595	7.1022	5.1463	5.1463	5.0375	5.0375	3.3020	3.3020	1.1850
		FOBT	6.6820	6.9307	4.8278	4.8278	4.9083	4.9083	3.2235	3.2235	1.1710
		HOBT	6.8082	6.9567	4.8800	4.8800	4.9283	4.9283	3.2358	3.2358	1.1735
		HOBT [15]	6.8087	-	-	-	-	-	-	3.2356	1.1734
2-2-1	0	CBT	12.1827	11.5140	8.4062	8.4062	8.2012	8.2012	5.3955	5.3955	1.9385
		FOBT	9.9977	11.0000	7.4652	7.4652	7.8070	7.8070	5.1525	5.1525	1.8945
		HOBT	10.0670	11.0020	7.4872	7.4872	7.8080	7.8080	5.1528	5.1528	1.8953
		HOBT [15]	10.0678	-	-	-	-	-	-	5.1528	1.8952
	1	CBT	9.2870	8.8185	6.4150	6.4075	6.2647	6.2647	4.1118	4.1118	1.4763
		FOBT	8.0325	8.5342	5.8892	5.8827	6.0492	6.0432	3.9785	3.9800	1.4528
		HOBT	8.1542	8.5577	5.9397	5.9332	6.0675	6.0615	4.0190	3.9913	1.4550
		HOBT [15]	8.1554	-	-	-	-	-	-	3.9896	1.4549
	10	CBT	7.0537	6.7002	4.8940	4.8665	4.7595	4.7595	3.1228	3.1228	1.1210
		FOBT	6.3322	6.5432	4.5960	4.5713	4.6405	4.6248	3.0460	3.0500	1.1083
		HOBT	6.4627	6.5707	4.6503	4.6253	4.6615	4.6455	3.1638	3.0630	1.1108
		HOBT [15]	6.4641	-	-	-	-	-	-	3.0588	1.1106

Table 6: Non-dimensional fundamental frequency of FG sandwich beams with FG core (Type C) for various BCs

(L/h=20, $\bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$)

Scheme	p	Theory	BCs								
			C–C	F–F	S–C	P–C	P–F	S–F	S–S	P–P	C–F
1-1-1	0	CBT	9.0180	8.9825	6.3320	6.2150	6.2050	6.2050	3.9795	3.9795	1.4185
		FOBT	8.9260	8.9600	6.2955	6.1800	6.1900	6.1850	3.9695	3.9710	1.4170
		HOBT	8.9375	8.9625	6.3000	6.1845	6.1920	6.1865	4.3870	3.9720	1.4175
	1	CBT	8.5240	8.4860	6.0590	5.8745	5.8640	5.8640	3.7615	3.7615	1.3410
		FOBT	8.4340	8.4640	6.0220	5.8405	5.8495	5.8400	3.7510	3.7535	1.3395
		HOBT	8.4375	8.4650	6.0235	5.8415	5.8500	5.8405	4.3825	3.7535	1.3395
	10	CBT	8.3600	8.3185	5.9815	5.7615	5.7500	5.7500	3.6890	3.6890	1.3155
		FOBT	8.2610	8.2940	5.9410	5.7240	5.7345	5.7210	3.6770	3.6805	1.3140
		HOBT	8.2460	8.2900	5.9345	5.7180	5.7315	5.7185	4.4140	3.6790	1.3135
1-2-1	0	CBT	9.6390	9.6030	6.7135	6.6430	6.6330	6.6330	4.2535	4.2535	1.5160
		FOBT	9.5330	9.5770	6.6725	6.6025	6.6155	6.6125	4.2430	4.2440	1.5145
		HOBT	9.5440	9.5795	6.6765	6.6070	6.6175	6.6140	4.5015	4.2445	1.5145
		HOBT[16]	9.5451	-	-	-	-	-	-	4.2445	1.5145
	1	CBT	8.7185	8.6805	6.1735	6.0090	5.9985	5.9985	3.8475	3.8475	1.3715
		FOBT	8.6225	8.6570	6.1350	5.9725	5.9830	5.9740	3.8365	3.8390	1.3700
		HOBT	8.6255	8.6575	6.1360	5.9735	5.9830	5.9745	4.4075	3.8390	1.3700
		HOBT[16]	8.6264	-	-	-	-	-	-	3.8387	1.3700
	10	CBT	8.4630	8.4195	6.0365	5.8325	5.8205	5.8205	3.7345	3.7345	1.3315
		FOBT	8.3505	8.3915	5.9905	5.7895	5.8025	5.7885	3.7210	3.7245	1.3300
		HOBT	8.3200	8.3835	5.9775	5.7775	5.7975	5.7835	4.4080	3.7215	1.3295
		HOBT[16]	8.3205	-	-	-	-	-	-	3.7214	1.3292
2-2-1	0	CBT	8.6585	8.6230	6.1220	5.9675	5.9570	5.9570	3.8205	3.8205	1.3620
		FOBT	8.5720	8.6015	6.0875	5.9345	5.9435	5.9360	3.8110	3.8130	1.3605
		HOBT	8.5825	8.6040	6.0910	5.9380	5.9450	5.9375	4.3505	3.8140	1.3610
		HOBT[16]	8.5832	-	-	-	-	-	-	3.8136	1.3607
	1	CBT	8.4460	8.4065	6.0185	5.8210	5.8100	5.8100	3.7270	3.7270	1.3290
		FOBT	8.3490	8.3825	5.9785	5.7840	5.7945	5.7830	3.7155	3.7185	1.3275
		HOBT	8.3435	8.3805	5.9760	5.7820	5.7935	5.7820	4.3860	3.7180	1.3275
		HOBT[16]	8.3442	-	-	-	-	-	-	3.7177	1.3271
	10	CBT	8.5460	8.5015	6.0680	5.8900	5.8775	5.8775	3.7715	3.7715	1.3445
		FOBT	8.4185	8.4700	6.0165	5.8410	5.8570	5.8435	3.7565	3.7600	1.3425
		HOBT	8.3730	8.4580	5.9975	5.8235	5.8495	5.8360	4.3625	3.7555	1.3420
		HOBT[16]	8.3738	-	-	-	-	-	-	3.7552	1.3418

Table 7: Non-dimensional fundamental natural frequency of FG sandwich beams with FG core (Type C) for various BCs (L/h=5, $\bar{\varpi}=\frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$)

Scheme	p	Theory	BCs								
			C–C	F–F	S–C	P–C	P–F	S–F	S–S	P–P	C–F
1-1-1	0	CBT	8.8465	8.3495	6.2125	6.1042	5.9560	5.9560	3.9183	3.9183	1.4080
		FOBT	7.7235	8.1055	5.7315	5.6365	5.7667	5.6860	3.7795	3.8015	1.3870
		HOBT	7.8667	8.1342	5.7920	5.6957	5.7885	5.7062	4.1948	3.8148	1.3898
	1	CBT	8.3410	7.8145	5.9277	5.7560	5.6015	5.6015	3.6963	3.6963	1.3298
		FOBT	7.2607	7.5875	5.4547	5.3050	5.4210	5.2848	3.5470	3.5835	1.3095
		HOBT	7.3270	7.5972	5.4785	5.3295	5.4278	5.2915	4.1513	3.5873	1.3105
	10	CBT	8.1582	7.5827	5.8360	5.6307	5.4640	5.4640	3.6173	3.6173	1.3030
		FOBT	7.0057	7.3465	5.3218	5.1453	5.2713	5.0835	3.4458	3.4950	1.2808
		HOBT	6.9060	7.3125	5.2623	5.0915	5.2423	5.0618	4.1120	3.4763	1.2778
1-2-1	0	CBT	9.4637	8.9542	6.5945	6.5297	6.3765	6.3765	4.1908	4.1908	1.5055
		FOBT	8.1847	8.6695	6.0510	5.9945	6.1577	6.1065	4.0428	4.0565	1.4813
		HOBT	8.3232	8.6965	6.1090	6.0515	6.1782	6.1255	4.3043	4.0690	1.4840
		HOBT[16]	8.3282	-	-	-	-	-	-	4.0691	1.4840
	1	CBT	8.5340	8.0012	6.0422	5.8892	5.7322	5.7322	3.7815	3.7815	1.3603
		FOBT	7.3862	7.7575	5.5415	5.4085	5.5392	5.4120	3.6273	3.6610	1.3385
		HOBT	7.4452	7.7647	5.5620	5.4293	5.5440	5.4170	4.1733	3.6638	1.3395
		HOBT[16]	7.4487	-	-	-	-	-	-	3.6636	1.3393
	10	CBT	8.2517	7.6497	5.8847	5.6955	5.5217	5.5217	3.6595	3.6595	1.3188
		FOBT	6.9715	7.3857	5.3123	5.1513	5.3048	5.1068	3.4705	3.5210	1.2933
		HOBT	6.7540	7.3165	5.1918	5.0383	5.2470	5.0615	4.0640	3.4840	1.2870
		HOBT[16]	6.7543	-	-	-	-	-	-	3.4830	1.2867
2-2-1	0	CBT	8.4857	7.9865	5.9995	5.8555	5.7077	5.7077	3.7593	3.7593	1.3515
		FOBT	7.4360	7.7625	5.5455	5.4190	5.5322	5.4268	3.6218	3.6503	1.3320
		HOBT	7.5655	7.7882	5.5997	5.4722	5.5517	5.4448	4.1515	3.6623	1.3343
		HOBT[16]	7.5709	-	-	-	-	-	-	3.6624	1.3344
	1	CBT	8.2522	7.6962	5.8790	5.6952	5.5332	5.5332	3.6580	3.6580	1.3170
		FOBT	7.1057	7.4577	5.3723	5.2133	5.3410	5.1780	3.4940	3.5368	1.2950
		HOBT	7.0882	7.4455	5.3548	5.1988	5.3300	5.1700	4.1170	3.5295	1.2940
		HOBT[16]	7.0901	-	-	-	-	-	-	3.5292	1.2939
	10	CBT	8.3300	7.7132	5.9142	5.7492	5.5717	5.5717	3.6945	3.6945	1.3315
		FOBT	6.9167	7.4175	5.2818	5.1433	5.3280	5.1315	3.4903	3.5388	1.3030
		HOBT	6.5982	7.3140	5.1075	4.9768	5.2430	5.0635	3.9873	3.4843	1.2933
		HOBT[16]	8.8465	8.3495	6.2125	6.1042	5.9560	5.9560	3.9183	3.9183	1.4080

Table 8: Non-dimensional fundamental frequency of FG beams with E-E boundary condition and various spring factors (β_R=β_T, L/h=10, $\bar{\omega} = \omega L\sqrt{\frac{\rho_m}{E_m}}$)

p	Theory	Spring stiffness										
		β_R=β_T	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^2	10^3	10^4	10^6
0	FOBT		0.0117	0.0369	0.1147	0.3121	0.5116	0.6751	0.9993	1.1431	1.1629	1.1652
	HOBT		0.0117	0.0369	0.1147	0.3122	0.5129	0.7063	1.0550	1.1539	1.1653	1.1665
0.5	FOBT	Present	0.0124	0.0390	0.1203	0.3097	0.4663	0.6279	0.8957	0.9857	0.9970	0.9982
		[26]	0.0123	0.0390	0.1202	0.3097	0.4663	0.6279	0.8957	0.9857	0.9970	0.9983
	HOBT		0.0124	0.0390	0.1203	0.3098	0.4681	0.6631	0.9331	0.9927	0.9993	1.0000
1	FOBT	Present	0.0127	0.0401	0.1233	0.3074	0.4455	0.6023	0.8299	0.8959	0.9039	0.9048
		[26]	0.0127	0.0401	0.1233	0.3074	0.4455	0.6023	0.8299	0.8959	0.9039	0.9048
	HOBT		0.0127	0.0402	0.1233	0.3075	0.4477	0.6390	0.8587	0.9007	0.9052	0.9057
5	FOBT	Present	0.0136	0.0429	0.1301	0.3005	0.4090	0.5624	0.7334	0.7725	0.7769	0.7774
		[26]	0.0136	0.0429	0.1301	0.3005	0.4090	0.5624	0.7334	0.7725	0.7769	0.7774
	HOBT		0.0137	0.0429	0.1301	0.3001	0.4125	0.6037	0.7458	0.7673	0.7696	0.7698
10	FOBT		0.0138	0.0436	0.1311	0.2912	0.3893	0.5486	0.7103	0.7448	0.7486	0.7490
	HOBT		0.0139	0.0436	0.1311	0.2908	0.3939	0.5920	0.7205	0.7390	0.7409	0.7411

Table 9: Non-dimensional fundamental frequency of FG and FG sandwich beams with various boundary conditions (β_R=β_T=10^2 for Elastic support, $\bar{\omega} = \omega L\sqrt{\frac{\rho_m}{E_m}}$)

Type	Scheme	P	Theory	L/h=5			L/h=20		
				C-E	E-E	H-E	C-E	E-E	H-E
Type A		0	FOBT	1.8768	1.7619	1.3909	0.5626	0.5189	0.3891
			HOBT	1.9816	1.8600	1.4331	0.5925	0.5455	0.3997
		0.5	FOBT	1.6571	1.5842	1.2309	0.4917	0.4646	0.3428
			[26]	1.6571	1.5842	1.2309	0.4917	0.4646	0.3428
			HOBT	1.7317	1.6537	1.2522	0.5136	0.4827	0.3474
		1	FOBT	1.5234	1.4697	1.1384	0.4501	0.4303	0.3167
			HOBT	1.5783	1.5225	1.1435	0.4671	0.4445	0.3171
		10	FOBT	1.2370	1.2117	0.9388	0.3842	0.3734	0.2699
			HOBT	1.2288	1.2077	0.9229	0.3933	0.3817	0.2699
Type B	1-1-1	0	FOBT	1.8768	1.7619	1.3909	0.5626	0.5189	0.3891
			HOBT	1.9816	1.8600	1.4331	0.5925	0.5455	0.3997
		1	FOBT	1.5062	1.4504	1.0978	0.4328	0.4138	0.2994
			HOBT	1.5791	1.5106	1.1233	0.4489	0.4251	0.3036
		10	FOBT	1.1701	1.1459	0.8419	0.3261	0.3185	0.2254
			HOBT	1.2200	1.1902	0.8603	0.3334	0.3239	0.2275
Type C	1-1-1	0	FOBT	1.4887	1.4359	1.1068	0.4270	0.4090	0.3023
			HOBT	1.5612	1.5035	1.1114	0.4420	0.4218	0.3001
		1	FOBT	1.4077	1.3657	1.0618	0.4061	0.3915	0.2917
			HOBT	1.4567	1.4144	1.0453	0.4188	0.4029	0.2853
		10	FOBT	1.3628	1.3267	1.0402	0.3990	0.3859	0.2889
			HOBT	1.3745	1.3416	1.0017	0.4101	0.3958	0.2800

Table 10: Non-dimensional fundamental frequency of non-rotational FG sandwich beams (1-1-1) with different translational spring factors ($\bar{\omega} = \omega L \sqrt{\dfrac{\rho_m}{E_m}}$)

Type	p	Theory	L/h=5					L/h=20				
			β_T=10^{-4}	1	10	10^2	10^6	β_T=10^{-4}	1	10	10^2	10^6
Type A	0	CBT	0.0117	1.0827	2.1171	2.4017	2.4366	0.0117	0.5643	0.6147	0.6201	0.6207
		FOBT	0.0117	1.0374	1.8056	1.9786	1.9995	0.0117	0.5569	0.6052	0.6104	0.6110
		HOBT	0.0117	1.0400	1.8171	1.9923	2.0134	0.0118	0.5570	0.6055	0.6106	0.6111
	1	CBT	0.0127	1.0974	1.7445	1.8595	1.8729	0.0127	0.4553	0.4755	0.4775	0.4778
		FOBT	0.0127	1.0297	1.4968	1.5713	1.5799	0.0127	0.4498	0.4692	0.4712	0.4715
		HOBT	0.0128	1.0332	1.5056	1.5809	1.5896	0.0128	0.4500	0.4693	0.4714	0.4715
	10	CBT	0.0139	1.0918	1.5103	1.5693	1.5759	0.0138	0.3913	0.4016	0.4027	0.4028
		FOBT	0.0139	0.9693	1.2259	1.2592	1.2630	0.0138	0.3845	0.3944	0.3954	0.3955
		HOBT	0.0139	0.9568	1.1984	1.2293	1.2328	0.0139	0.3834	0.3932	0.3942	0.3943
Type B	0	CBT	0.0117	1.0827	2.1171	2.4017	2.4366	0.0117	0.5643	0.6147	0.6201	0.6207
		FOBT	0.0117	1.0374	1.8056	1.9786	1.9995	0.0117	0.5569	0.6052	0.6104	0.6110
		HOBT	0.0117	1.0400	1.8171	1.9923	2.0134	0.0118	0.5570	0.6055	0.6106	0.6111
	1	CBT	0.0124	1.0611	1.6787	1.7881	1.8008	0.0124	0.4369	0.4557	0.4576	0.4579
		FOBT	0.0124	1.0072	1.4794	1.5558	1.5647	0.0124	0.4326	0.4509	0.4528	0.4530
		HOBT	0.0124	1.0146	1.5027	1.5822	1.5914	0.0125	0.4332	0.4515	0.4535	0.4536
	10	CBT	0.0130	0.9731	1.2792	1.3195	1.3241	0.0130	0.3289	0.3357	0.3364	0.3365
		FOBT	0.0130	0.9176	1.1605	1.1916	1.1951	0.0130	0.3265	0.3332	0.3339	0.3340
		HOBT	0.0130	0.9313	1.1880	1.2210	1.2247	0.0130	0.3271	0.3339	0.3346	0.3347
Type C	0	CBT	0.0124	1.0561	1.6538	1.7573	1.7693	0.0124	0.4308	0.4488	0.4507	0.4509
		FOBT	0.0124	1.0029	1.4629	1.5362	1.5446	0.0124	0.4267	0.4443	0.4461	0.4463
		HOBT	0.0124	1.0111	1.4879	1.5645	1.5734	0.0125	0.4274	0.4449	0.4468	0.4470
	1	CBT	0.0127	1.0590	1.5768	1.6588	1.6682	0.0127	0.4102	0.4245	0.4260	0.4262
		FOBT	0.0127	0.9971	1.3879	1.4455	1.4521	0.0127	0.4061	0.4201	0.4215	0.4217
		HOBT	0.0128	1.0020	1.3999	1.4587	1.4654	0.0128	0.4063	0.4204	0.4218	0.4220
	10	CBT	0.0131	1.0695	1.5507	1.6234	1.6317	0.0131	0.4036	0.4165	0.4178	0.4180
		FOBT	0.0131	0.9959	1.3460	1.3955	1.4011	0.0131	0.3991	0.4116	0.4129	0.4130
		HOBT	0.0131	0.9897	1.3285	1.3758	1.3812	0.0132	0.3985	0.4109	0.4123	0.4124

Table 11: Non-dimensional fundamental frequency of non-translational FG sandwich beams (1-1-1) with different rotational spring factors ($\bar{\omega} = \omega L \sqrt{\frac{\rho_m}{E_m}}$)

Ty	p	Theory	L/h=5					L/h=20				
			$\beta_R=10^{-4}$	1	10	10^2	10^6	$\beta_R=10^{-4}$	1	10	10^2	10^6
Type A	0	FOBT	1.0305	1.0642	1.2820	1.7770	1.9995	0.2730	0.2828	0.3484	0.5193	0.6110
		HOBT	1.0306	1.0723	1.3752	1.8774	2.0134	0.2730	0.2832	0.3582	0.5458	0.6111
	1	FOBT	0.8510	0.8952	1.1316	1.4769	1.5799	0.2251	0.2379	0.3099	0.4305	0.4715
		HOBT	0.8511	0.9071	1.2251	1.5304	1.5896	0.2251	0.2385	0.3227	0.4447	0.4715
	10	FOBT	0.6974	0.7554	0.9944	1.2151	1.2630	0.1871	0.2047	0.2845	0.3735	0.3955
		HOBT	0.6903	0.7712	1.0650	1.2110	1.2328	0.1870	0.2062	0.3039	0.3818	0.3943
Type B	0	FOBT	1.0305	1.0642	1.2820	1.7770	1.9995	0.2730	0.2828	0.3484	0.5193	0.6110
		HOBT	1.0306	1.0723	1.3752	1.8774	2.0134	0.2730	0.2832	0.3582	0.5458	0.6111
	1	FOBT	0.7725	0.8241	1.0894	1.4577	1.5647	0.2016	0.2159	0.2932	0.4139	0.4530
		HOBT	0.7751	0.8364	1.1728	1.5187	1.5914	0.2018	0.2165	0.3032	0.4253	0.4536
	10	FOBT	0.5732	0.6476	0.9232	1.1490	1.1951	0.1483	0.1685	0.2474	0.3186	0.3340
		HOBT	0.5762	0.6637	0.9974	1.1937	1.2247	0.1484	0.1693	0.2568	0.3240	0.3347
Type C	0	FOBT	0.8355	0.8777	1.1056	1.4429	1.5446	0.2193	0.2312	0.2983	0.4092	0.4463
		HOBT	0.8389	0.8911	1.1973	1.5114	1.5734	0.2195	0.2318	0.3088	0.4221	0.4470
	1	FOBT	0.8301	0.8709	1.0830	1.3714	1.4521	0.2191	0.2308	0.2946	0.3916	0.4217
		HOBT	0.8303	0.8842	1.1723	1.4206	1.4654	0.2192	0.2315	0.3068	0.4030	0.4220
	10	FOBT	0.8303	0.8701	1.0720	1.3316	1.4011	0.2208	0.2326	0.2951	0.3860	0.4130
		HOBT	0.8224	0.8777	1.1455	1.3467	1.3812	0.2208	0.2333	0.3084	0.3959	0.4124

CAPTIONS OF FIGURES

Fig. 1: Geometry FG sandwich beams.

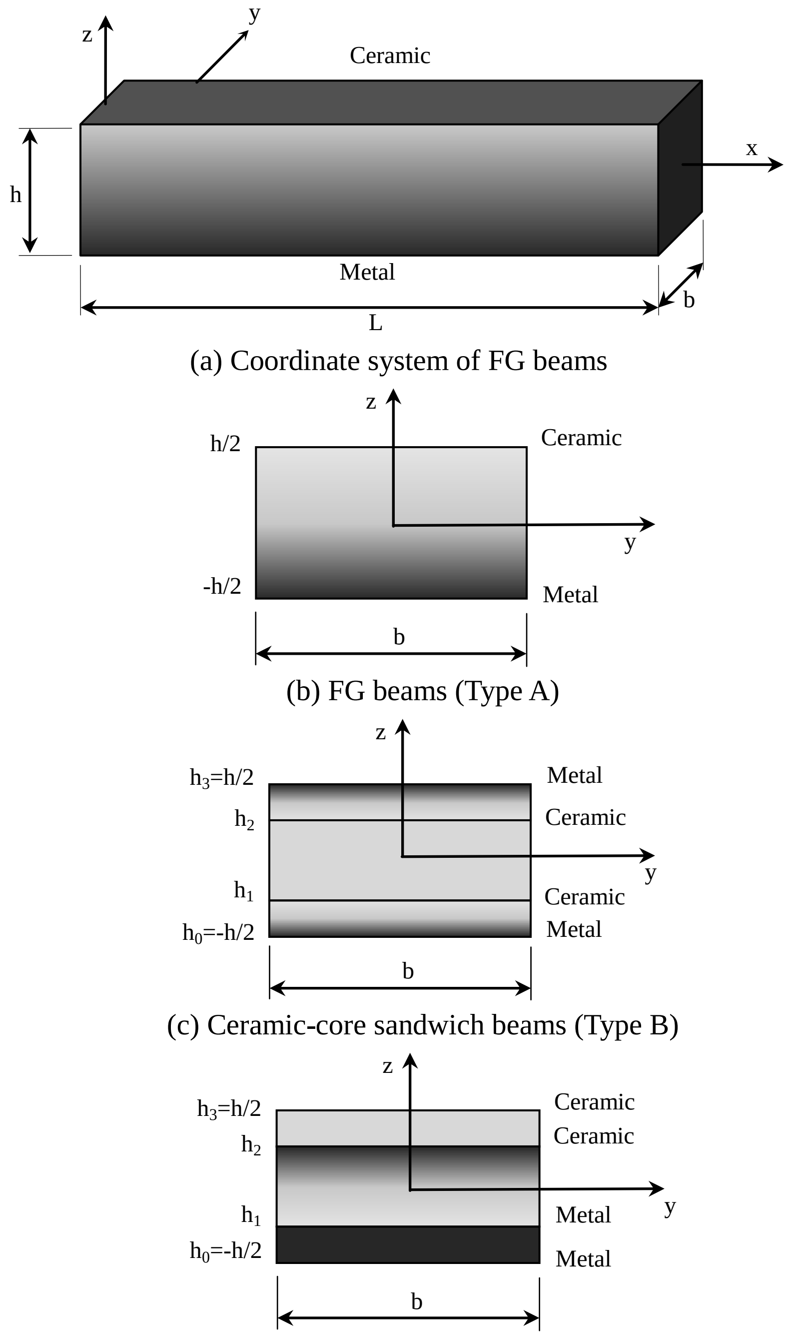
Fig. 2: Variation of Young's modulus $E(z)$ through the beam depth according to the power-law form.

Fig. 3: Classical and non-classical boundary conditions.

Fig. 4: Non-dimensional fundamental frequency of non-rotational ($\beta_R=10^6$) FG beams with various translational spring factors (L/h=10, p=0.5, $\bar{\omega}=\omega L\sqrt{\frac{\rho_m}{E_m}}$)

Fig. 5: Non-dimensional fundamental frequency of non-translational ($\beta_T=10^6$) FG beams with various rotational spring factors (L/h=10, p=0.5, $\bar{\omega}=\omega L\sqrt{\frac{\rho_m}{E_m}}$)

Fig. 6: Non-dimensional fundamental frequency of FG beams with various rotational and translational spring factors (L/h=10, p=1, $\bar{\omega}=\omega L\sqrt{\frac{\rho_m}{E_m}}$)



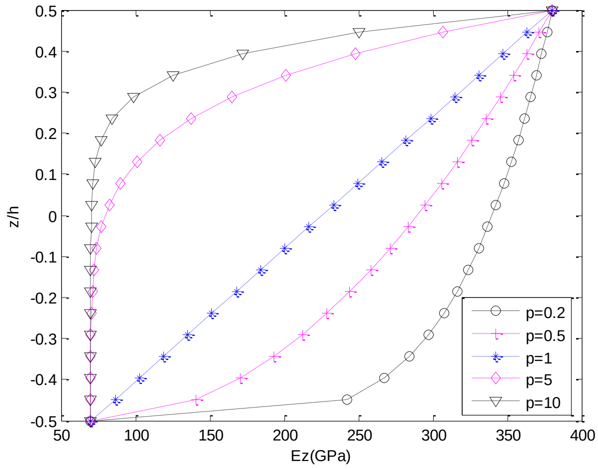
(a) Coordinate system of FG beams

(b) FG beams (Type A)

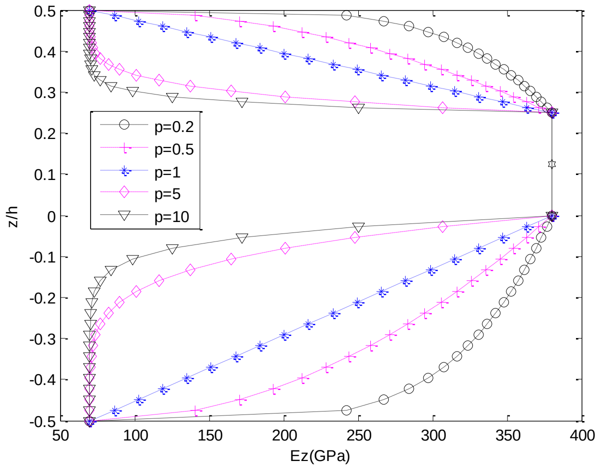
(c) Ceramic-core sandwich beams (Type B)

(d) FG-core sandwich beams (Type C)

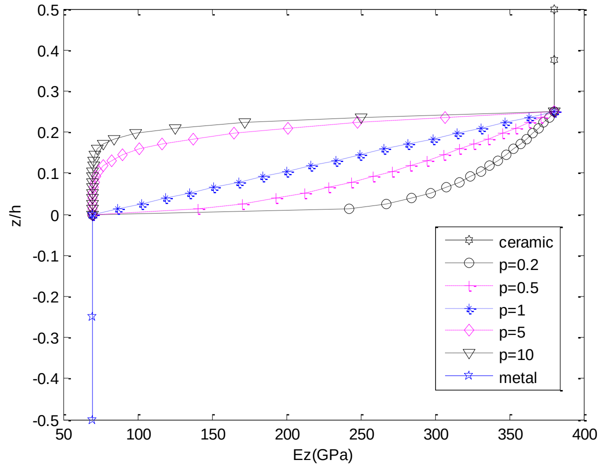
Fig. 1: Geometry of FG sandwich beams.



(a) Type A



(b) Type B (2-1-1)



(c) Type C (2-1-1)

Fig. 2: Variation of Young's modulus $E(z)$ through the beam depth according to the power-law form.

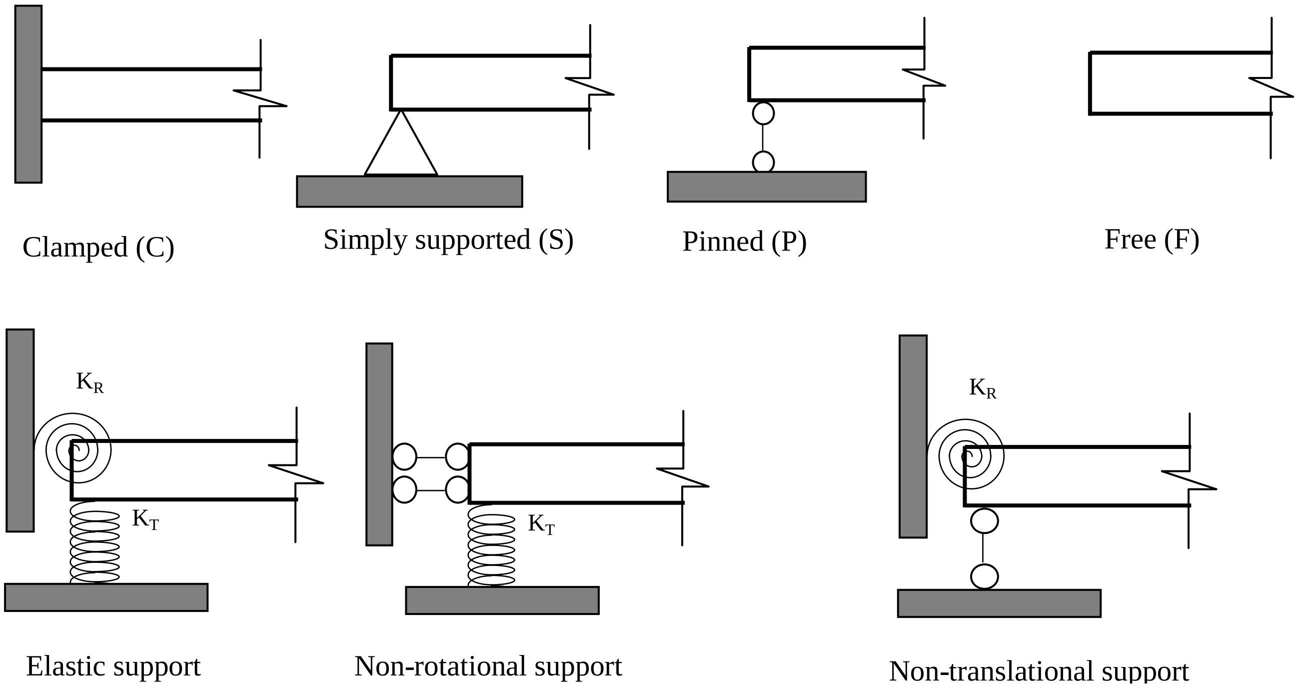


Fig. 3: Classical and non-classical boundary conditions.

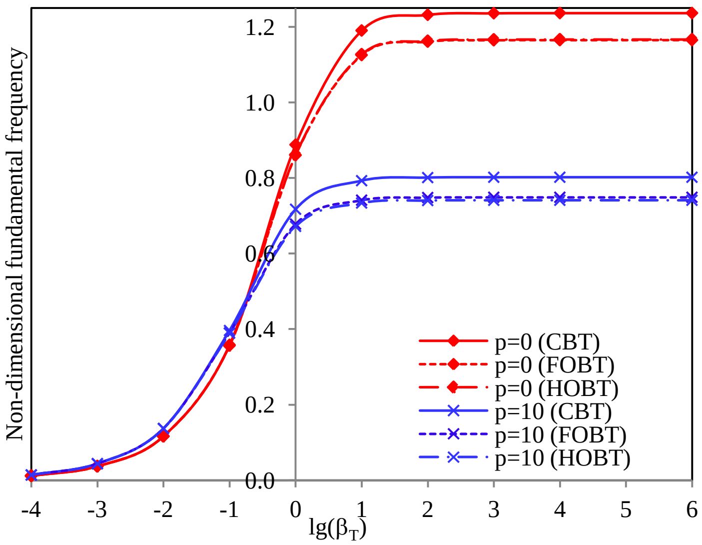


Fig. 4: Non-dimensional fundamental frequency of non-rotational ($\beta_R=10^6$) FG beams with various translational spring factors (L/h=10, p=0.5, $\bar{\omega}=\omega L\sqrt{\frac{\rho_m}{E_m}}$)

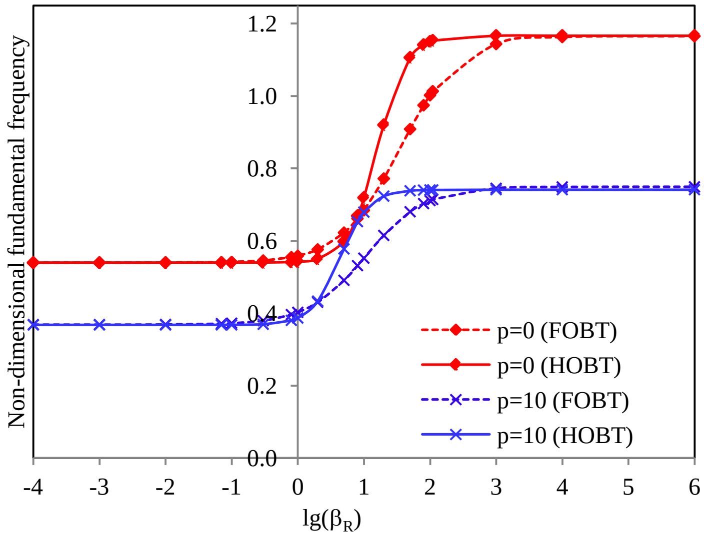


Fig. 5: Non-dimensional fundamental frequency of non-translational ($\beta_T=10^6$) FG beams with various rotational spring factors (L/h=10, p=0.5, $\bar{\omega}=\omega L\sqrt{\frac{\rho_m}{E_m}}$)

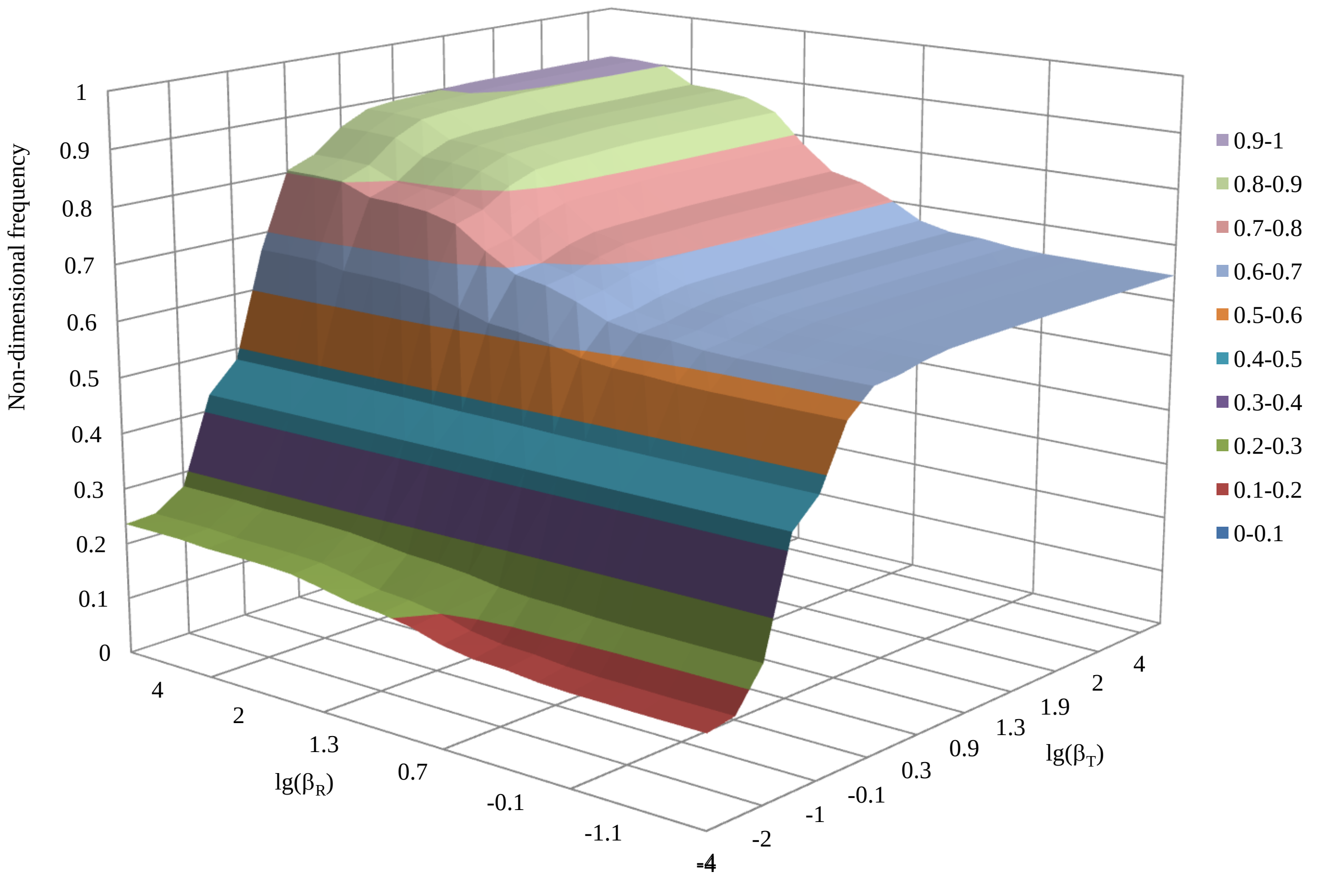


Fig. 6: Non-dimensional fundamental frequency of FG beams with various rotational and translational spring factors (L/h=10, p=1, $\bar{\omega} = \omega L\sqrt{\frac{\rho_m}{E_m}}$)